Algebra III
Back Paper Exam
December 2004

Instructions. Attempt **all** questions. The field of rational numbers is denoted by $\mathbb{Q}$.

1. (a) Identify the group $\mathbb{F}_4^+$. (3)

   (b) Let p be a prime. Describe the integers n such that there exist a finite field K of order n and an element in the multiplicative group $K^\times$ whose order is p. (3)

   (c) Let p be a prime number. Find the number of monic irreducible polynomials of degree 2 in the ring $\mathbb{F}_p[X]$. (4)

2. (a) Let F be a field of characteristic zero. Let f' denote the derivative of a polynomial $f \in F[X]$, and let g be an irreducible polynomial which is a common divisor of f and f'. Prove that g^2 divides f. (5)

   (b) Let α be a complex root of the irreducible polynomial $X^3 - 3X + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in $\mathbb{Q}(\alpha)$ explicitely, in the form $a + b\alpha + c\alpha^2$, with $a, b, c \in \mathbb{Q}$. (5)

3. (a) Let β, γ be complex numbers of degree 3 over $\mathbb{Q}$, and let $K = \mathbb{Q}(\beta, \gamma)$. Determine the possibilities of $[K : \mathbb{Q}]$. (3)

   (b) Let ζ, η be complex roots of irreducible polynomials $f(X), g(X) \in \mathbb{Q}[X]$ respectively. Let $F = \mathbb{Q}(\zeta)$ and $K = \mathbb{Q}(\eta)$. Prove that $f(X)$ is irreducible in $K[X]$ if and only if $g(X)$ is irreducible in $F[X]$. (3)

   (c) Determine the irreducible polynomial of $\sqrt{3} + \sqrt{5}$ over $\mathbb{Q}(\sqrt{5})$ and over $\mathbb{Q}(\sqrt{15})$. (4)

4. (a) Prove that every Galois extension K over F whose Galois group is the Klein four group is biquadratic. (4)

   (b) Let C_n denote the cyclic group of order n. Let K be a Galois extension of a field F such that $G(K/F) = C_2 \times C_{12}$. How many intermediate fields L are there such that

   1. $[L : F] = 4$,
   2. $[L : F] = 9$
   3. $G(K/L) \approx C_4$ (6)

5. (a) Let $F \subset L \subset K$ be fields. Prove or disprove:

   1. If K/F is Galois, then K/L is Galois.
   2. If K/F is Galois, then L/F is Galois.
   3. If L/F is Galois and K/L is Galois, then K/F is Galois. (6)

   (b) Let G be a finite group. Prove that there exists a field F and a Galois extension K of F whose Galois group is G. (4)

6. (a) Let K be a finite extension of $\mathbb{Q}$. Prove that there exists an element $\xi \in K$ such that $K = \mathbb{Q}(\xi)$. (5)

   (b) Let L be the splitting field of the polynomial $X^3 + X + 1$ over $\mathbb{Q}$. Find an element $\theta \in L$ such that $L = \mathbb{Q}(\theta)$. (2)

   (c) Let p and q be prime numbers and K be the splitting field of $X^p - q$ over $\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = p(p-1)$. (3)